Data-Driven Estimation of Quadratic Transfer Functions using NARX and Harmonic Probing

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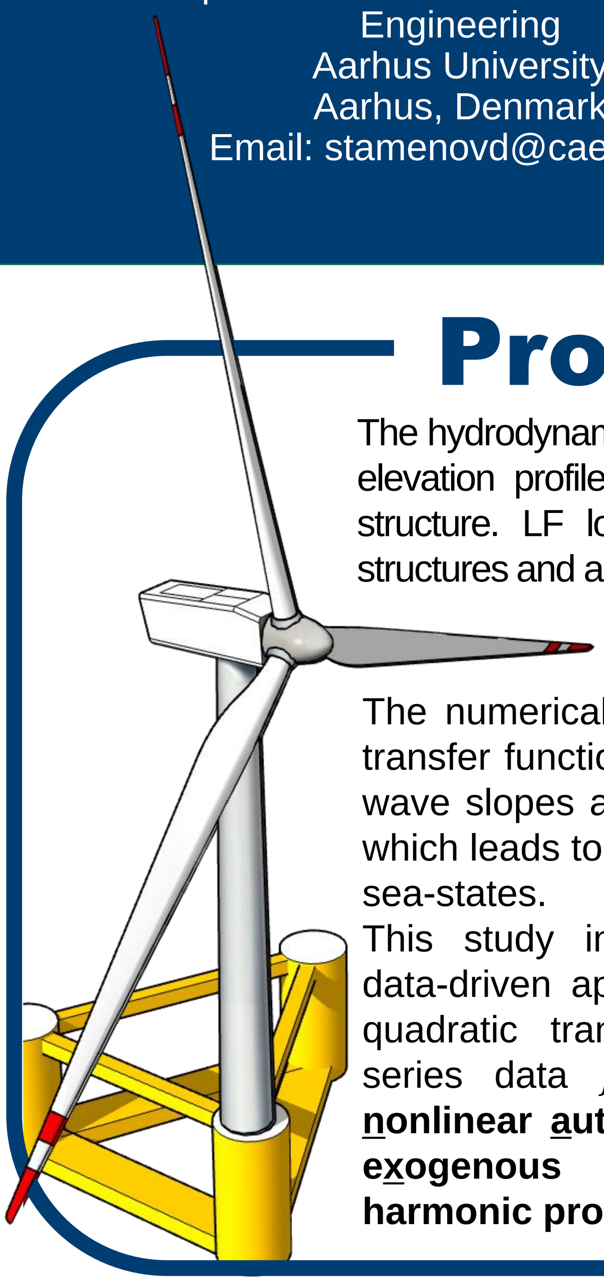
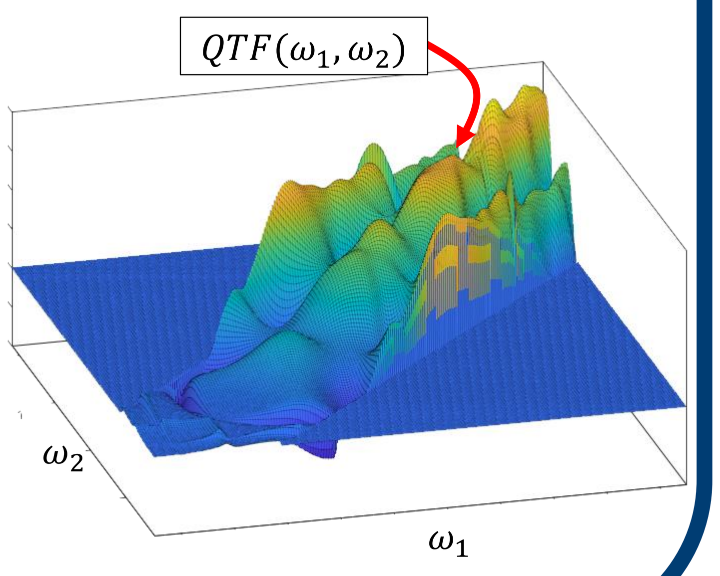
Problem Statement



The hydrodynamic low-frequency (LF) loading, $f(t)$, is a nonlinear functional of the wave elevation profile, $\zeta(t)$, and depends on the hydrodynamic properties of the floating structure. LF loads become prominent in determining extreme offsets in moored structures and are generally described by a Quadratic Transfer Function (QTF).

The numerical tools for estimating such transfer functions assume infinitely small wave slopes and neglect viscous effects which leads to inaccurate results in harsh sea-states.

This study implements an alternative data-driven approach for estimating the quadratic transfer function from time series data $f(t)$ and $\zeta(t)$ using a **nonlinear auto-regressive model with exogenous input (NARX)** and **harmonic probing (HP)**.



Volterra Series

The Volterra series represents a nonlinear functional relationship between the input and output of a systems,

$$f(t) = f_1(t) + f_2(t) + \cdots + f_m(t) \quad (1)$$

Which can be conveniently expressed in the frequency domain as,

$$F(\omega) = F_1(\omega) + F_2(\omega) + \cdots + F_m(\omega) \quad (2)$$

$$F_1(\omega) = \int_{-\infty}^{+\infty} \delta(\omega - \omega_1) H^{(1)}(\omega_1) Z(\omega_1) d\omega_1 \quad (3)$$

$$F_2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(\omega - \omega_1 - \omega_2) H^{(2)}(\omega_1, \omega_2) Z(\omega_1) Z(\omega_2) d\omega_1 d\omega_2 \quad (4)$$

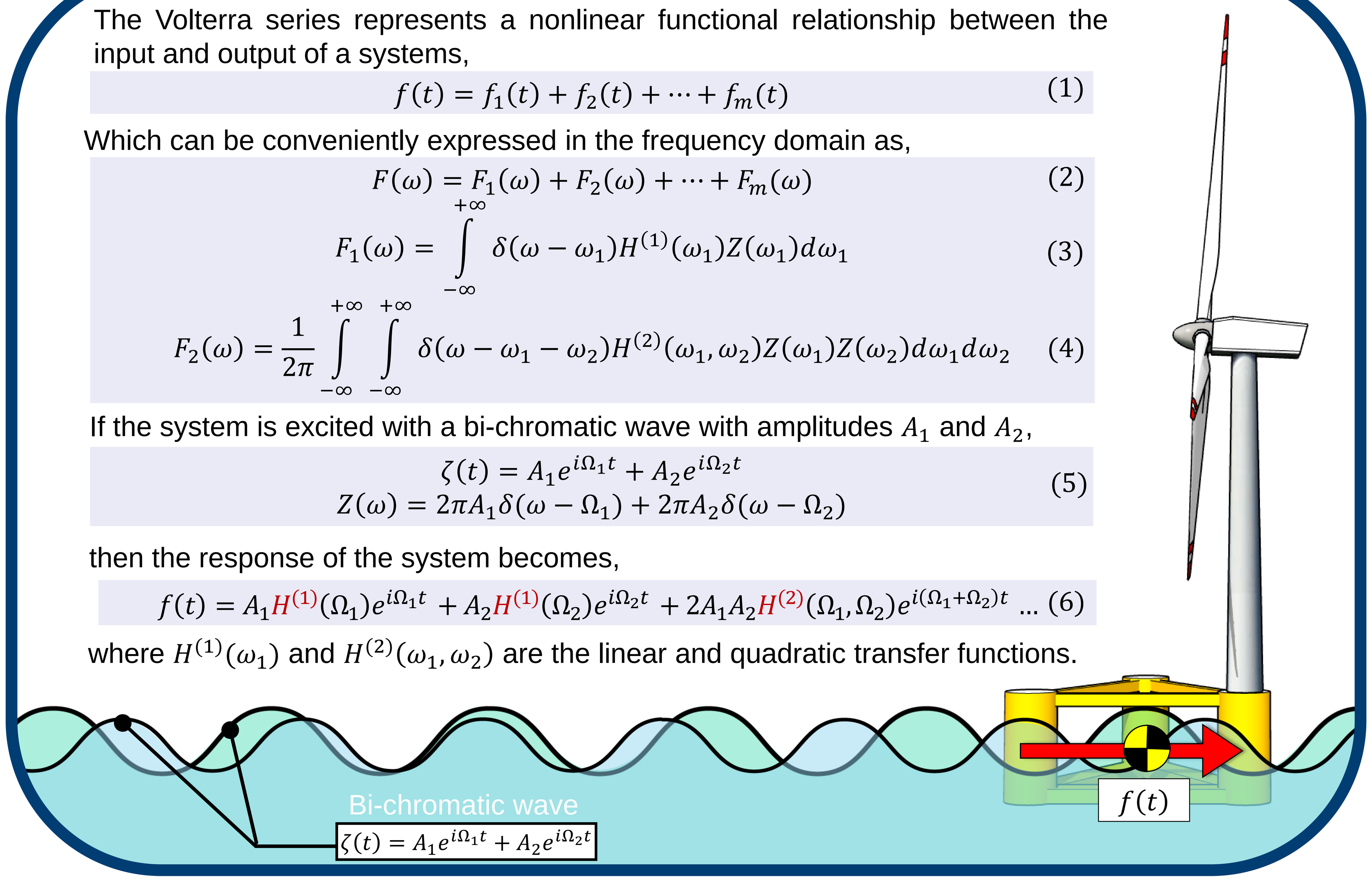
If the system is excited with a bi-chromatic wave with amplitudes A_1 and A_2,

$$\zeta(t) = A_1 e^{i\Omega_1 t} + A_2 e^{i\Omega_2 t}$$
$$Z(\omega) = 2\pi A_1 \delta(\omega - \Omega_1) + 2\pi A_2 \delta(\omega - \Omega_2) \quad (5)$$

then the response of the system becomes,

$$f(t) = A_1 H^{(1)}(\Omega_1) e^{i\Omega_1 t} + A_2 H^{(1)}(\Omega_2) e^{i\Omega_2 t} + 2A_1A_2 H^{(2)}(\Omega_1, \Omega_2) e^{i(\Omega_1 + \Omega_2)t} \ldots \quad (6)$$

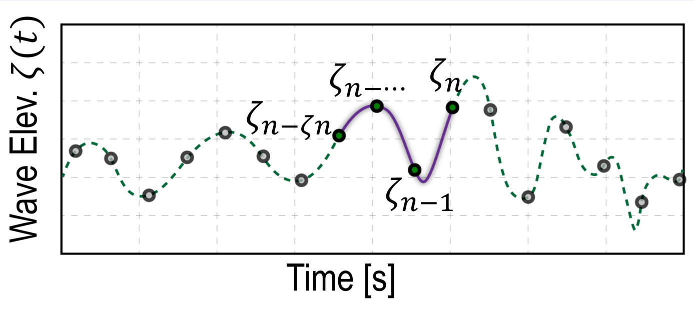
where $H^{(1)}(\omega_1)$ and $H^{(2)}(\omega_1, \omega_2)$ are the linear and quadratic transfer functions.

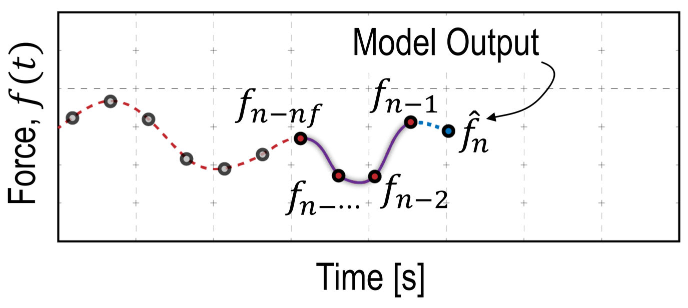


NARX Model

NARX is an autoregressive, forecasting, data-driven model. The key idea is that the next step prediction ($\hat{f}_n$) of the hydrodynamic force can be represented as a nonlinear function of a few past values of that same force (autoregressive part) and a wave elevation profile (exogenous part):

$$\hat{f}_n = \mathcal{F}(f_{n-1}, f_{n-2}, \ldots, f_{n-nf}, \zeta_n, \zeta_{n-1}, \ldots, \zeta_{n-\zeta n}) = \mathcal{F}(\boldsymbol{x}_n) \quad (7)$$





For this application we chose a polynomial NARX model whose form is shown below,

$$\hat{f}_n = \begin{bmatrix} C^{(1)}_1 & C^{(1)}_2 & C^{(1)}_3 & C^{(1)}_4 & C^{(1)}_{...} & C^{(1)}_p \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-...} \\ f_{n-nf} \\ \zeta_n \\ \zeta_{n-...} \\ \zeta_{n-n\zeta} \end{bmatrix} + \begin{bmatrix} f_{n-1} \\ f_{n-...} \\ f_{n-nf} \\ \zeta_n \\ \zeta_{n-...} \\ \zeta_{n-n\zeta} \end{bmatrix}^T \begin{bmatrix} C^{(2)}_{11} & C^{(2)}_{12} & C^{(2)}_{13} & C^{(2)}_{14} & C^{(2)}_{1...} & C^{(2)}_{1p} \\ C^{(2)}_{21} & C^{(2)}_{22} & C^{(2)}_{23} & C^{(2)}_{24} & C^{(2)}_{2...} & C^{(2)}_{2p} \\ C^{(2)}_{31} & C^{(2)}_{32} & C^{(2)}_{33} & C^{(2)}_{34} & C^{(2)}_{3...} & C^{(2)}_{3p} \\ C^{(2)}_{41} & C^{(2)}_{42} & C^{(2)}_{43} & C^{(2)}_{44} & C^{(2)}_{4...} & C^{(2)}_{4p} \\ C^{(2)}_{...1} & C^{(2)}_{...2} & C^{(2)}_{...3} & C^{(2)}_{...4} & C^{(2)}_{...} & C^{(2)}_{...p} \\ C^{(2)}_{p1} & C^{(2)}_{p2} & C^{(2)}_{p3} & C^{(2)}_{p4} & C^{(2)}_{p...} & C^{(2)}_{pp} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-...} \\ f_{n-nf} \\ \zeta_n \\ \zeta_{n-...} \\ \zeta_{n-n\zeta} \end{bmatrix}$$

The simple form of this model allows for an efficient training using an off-the-shelf regression algorithm. The training of the unknown coefficients is performed as follows,

$$(\boldsymbol{C}^{(1)}, \boldsymbol{C}^{(2)}) = \arg\min_{\boldsymbol{C}^{(1)}, \boldsymbol{C}^{(2)}} \left\{ \frac{1}{2N} \sum_{i=1}^{N} \left(f_i - \hat{f}_i(\boldsymbol{C}^{(1)}, \boldsymbol{C}^{(2)}) \right)^2 + \lambda \sum_{j=1}^{p} |C^{(1)}_j| + \lambda \sum_{k=1}^{p^2} |C^{(2)}_k| \right\} \quad (8)$$

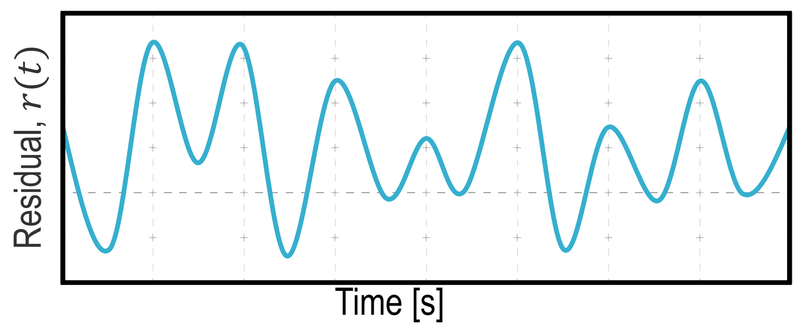
Harmonic Probing

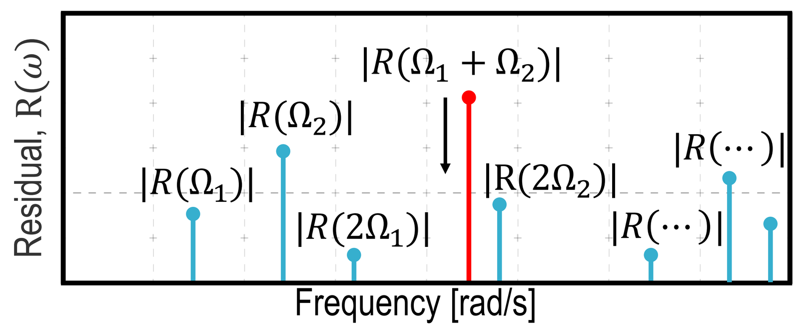
The idea behind harmonic probing (HP) is to equate the two loading models, namely, the *Volterra Series expansion* and *polynomial-NARX*. This allows for the, otherwise meaningless, NARX coefficients to be related to a physical quantity such as the transfer functions contained in the Volterra Series. This can be achieved by formulating a residual equation:

Residual = Volterra Output − NARX Output

$$r\left(\hat{H}^{(2)}(\Omega_1, \Omega_2), H^{(1)}(\Omega_1), H^{(1)}(\Omega_2)\right) = f(t) - [\mathbf{C}^{(1)} \boldsymbol{x}^T + \boldsymbol{x}\, \mathbf{C}^{(2)} \boldsymbol{x}^T] \quad (9)$$

Transforming the residual to the frequency domain gives us access to specific frequency bins whose amplitude is a function of the unknown transfer function:

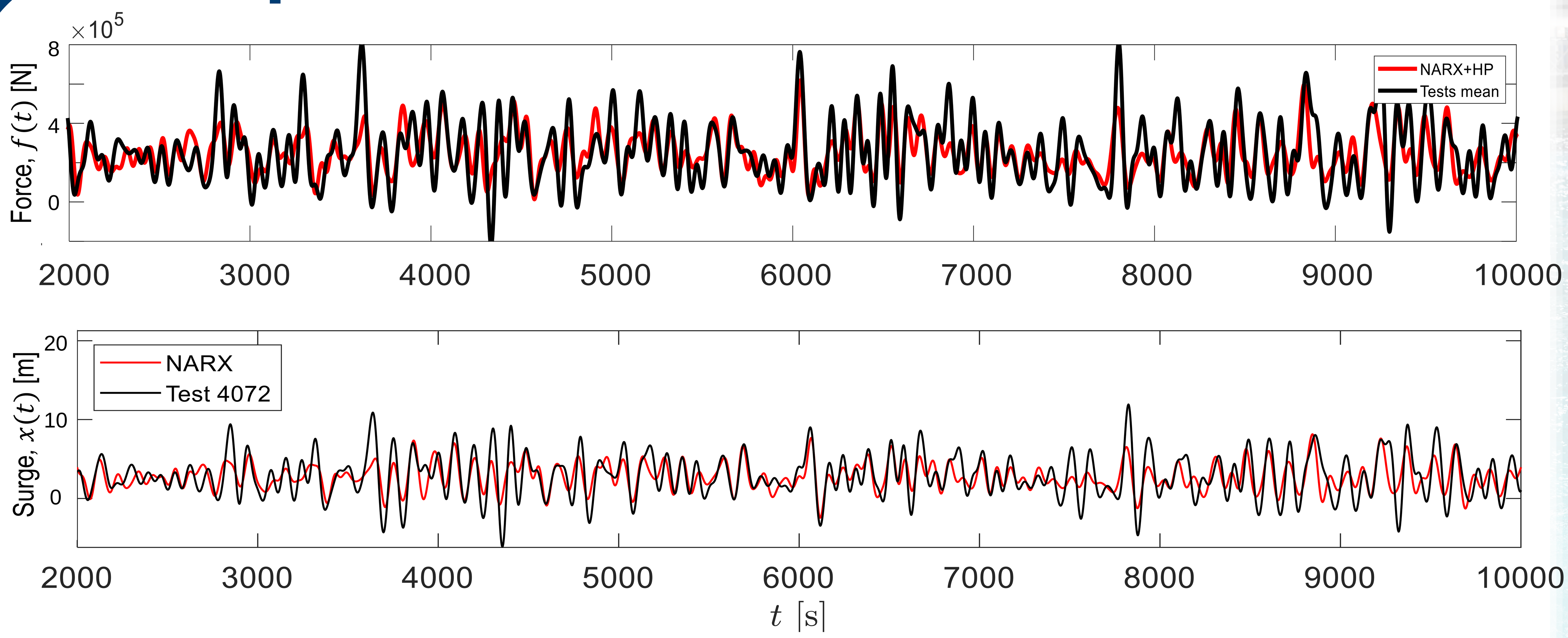




Minimizing the amplitude of the residual at $(\Omega_1 + \Omega_2)$ w.r.t $\hat{H}^{(2)}(\Omega_1, \Omega_2)$ yields the transfer function:

$$H^{(2)}(\Omega_1, \Omega_2) = \arg\min_{\hat{H}^{(2)}(\Omega_1, \Omega_2)} |R(\Omega_1 + \Omega_2)| \quad (10)$$

Experimental Verification



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