FEM formulations for nonlinear dynamics of shear- and torsion-free rods in mooring line applications

Thi-Hoa Nguyen[1], Bruno A. Roccia[1], Dominik Schillinger[2], Cristian G. Gebhardt[1]

[1]Geophysical Institute and Bergen Offshore Wind Centre, University of Bergen, Norway,
[2]Institute for Mechanics, Computational Mechanics Group, Technical University of Darmstadt, Germany

Introduction

In offshore and coastal engineering, long and slender cables or rods that are employed for towing or mooring of floating structures, belong to the most important research topics. In this work, we consider the shear- and torsion-free nonlinear Kirchhoff rod formulation developed in [1] for such applications. To solve the corresponding governing equations, nodal and isogeometric finite elements have been employed in [1] and [2], respectively.
In this work, we give an overview and attempt to gain a deeper understanding of the nodal and isogeomtric discretization schemes for the rod formulation [1]. We discuss the space of the resulting discrete solution, which is in multiple copies either of the manifold $\mathbb{R}^3 \times S^2$ or the Euclidean space $\mathbb{R}^3$. We compare the semi-discrete formulation, matrix equations, and computational cost of each discretization variant, and illustrate our findings via numerical examples of cables commonly employed as mooring lines.

An exemplary mooring line

- A mooring line of initial length of 627 m [5, p.257].
- Logarithmic current speed profile.
- 40 discrete elements and 4 iterations required at each time step.

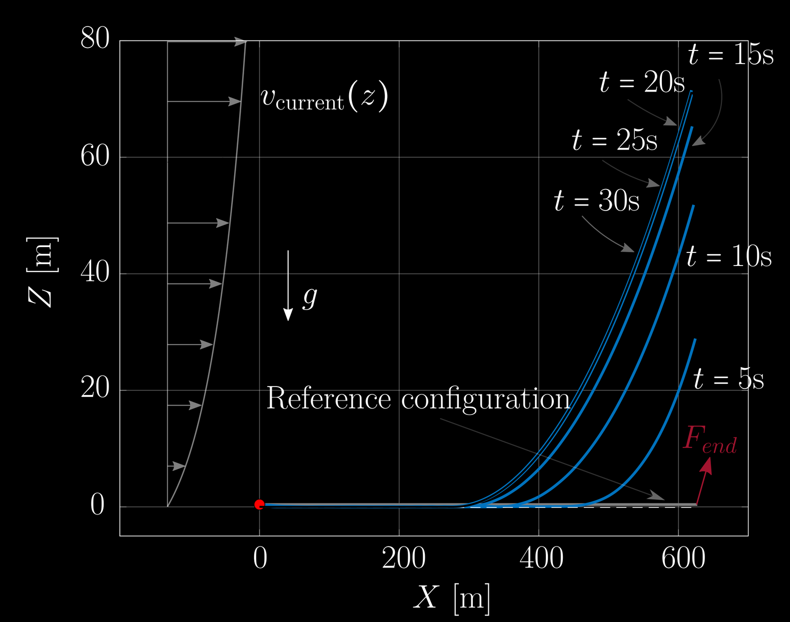

Figure 1: Snapshots of the studied mooring line, computed with cubic C^1 isogeometric discretization.

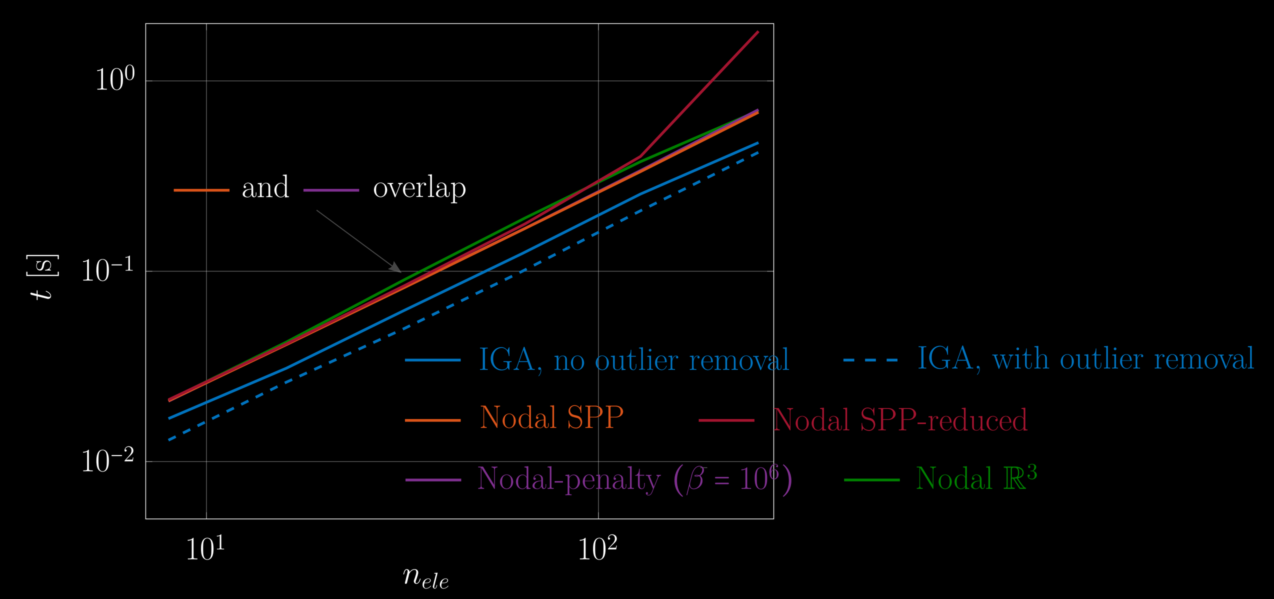

Figure 2: Averaged computing time per iteration and time step.

Conclusions and outlook

Discretizing the rod formulation [1] using nodal finite elements and preserving the unit sphere structure for the nodal directors leads to zero nodal axial stress values. We studied five discretization variants which generally lead to the same final rod configurations. For the studied benchmark, using the nodal scheme generally leads to better accuracy in the deformations. Nevertheless, using isogeometric discretizations requires less computation time per iteration and time step.
Future work includes investigations and elimination of membrane locking on the studied semi-discrete rod formulations and development of other strain measures that tackle zero nodal stress values.

Nonlinear rod formulation

Variational formulation in a continuous setting

Consider the following set for the rod configurations [1]:

$$\mathcal{D} := \left\{ \boldsymbol{\varphi} \in C^2\left([0,L], \mathbb{R}^3\right),\, |\boldsymbol{\varphi}'| > 0, \boldsymbol{\varphi}(0,t) = \mathbf{0},\, \boldsymbol{\varphi}'(0,t) = \boldsymbol{E}_3 \right\},$$

where $C^2[0,L]$ is the space of C^2 continuous functions on $[0,L]$, $\boldsymbol{\varphi} = \boldsymbol{\varphi}(s,t)$, $(s,t) \in [0,L] \times [0,T]$ is the configuration of Kirchhoff rods that are initially straight, shear-, torsion-free, transversely isotropic, and depend on the arc-length s and time t.
The weak formulation of the studied rods [1] is:

$$\int_0^S \delta\boldsymbol{\varphi} \cdot \left(\mathcal{M}(\boldsymbol{\varphi}')\, \hat{\nabla}_{\dot{\varphi}} \dot{\boldsymbol{\varphi}} + \mathcal{B}(\boldsymbol{\varphi}', \boldsymbol{\varphi}'')^T \boldsymbol{\sigma} - \boldsymbol{f}^{\mathrm{ext}} \right) \mathrm{d}s = 0\,.$$

For more details on the derivation of the weak form and more discussions, we refer to [1].

Spatial discretization

The studied rod formulation [1] requires at least C^1-continuity. One can spatially discretize $\boldsymbol{\varphi}(s,t) \in \mathcal{D}$ using isogeometric finite elements [2] as follows:

$$\boldsymbol{\varphi}(s,t) \approx \boldsymbol{\varphi}_h(s,t) = \sum_i^m B_i(s)\, \boldsymbol{x}_i(t) = \mathbf{B}(s)\, \boldsymbol{q}\,,$$

where B_i, $1 \le i \le m$ denotes smooth spline basis functions of degree p and conitnuity C^r, $1 \le r \le p-1$, and $\boldsymbol{q} = \boldsymbol{q}(t) \in (\mathbb{R}^3)^m$ is the vector of unknown time-dependent coefficients.
Alternatively, one can spatially discretize $\boldsymbol{\varphi}(s,t)$ using nodal finite elements based on cubic Hermite spline functions [1] as follows:

$$\boldsymbol{\varphi}_h(s,t) = \sum_{e=1}^{n_e} (H_1 \boldsymbol{x}_1^e + H_2 \boldsymbol{d}_1^e + H_3 \boldsymbol{x}_2^e + H_4 \boldsymbol{d}_2^e) = \mathbf{H}(s)\, \bar{\boldsymbol{q}}\,,$$

where H_i, $1 \le i \le 4$, is the standard cubic Hermite spline function, $\boldsymbol{x}_j^e \in \mathbb{R}^3$ and $\boldsymbol{d}_j^e \in S^2$, $j = 1,2$, is the nodal position and director at the j-th node of the e-th element, $1 \le e \le n_e$, respectively. Here n_e denotes the number of elements and $\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(t) \in (\mathbb{R}^3 \times S^2)^{n_n}$ the vector of unknown time-dependent coefficients, where $n_n = n_e + 1$ is the number of nodes. Since the defined director field [1] belongs to the unit sphere S^2, preserving this structure at the nodes requires an additional constraint of unit nodal directors:

$$\boldsymbol{d}_j^e \cdot \boldsymbol{d}_j^e = 1\,, \quad 1 \le e \le n_e\,,\ j = 1,2\,,$$

which can be enforced either strongly using the Lagrange multiplier method or weakly using the penalty method. Furthermore, when using the Lagrange multiplier method, one can eliminate the additional variable field of the Lagrange multipliers using the nullspace method. For more details on this, the resulting semi-discrete formulation, and matrix equations, we refer to [3].

<table>
<tr><th colspan="2">Discretization scheme</th><th>Discrete solution</th><th>System matrix</th><th>n_{dof}</th></tr>
<tr><td colspan="2">Isogeometric discretizations</td><td>$\mathbb{R}^{3m}$</td><td>sparse, symmetric$^{(1)}$</td><td>$3[n_e(p-r)+r+1]$</td></tr>
<tr><td colspan="5">Note that the axial stress resultant is not constrained to zero at any point and the discrete director field lives in S^2 at any point of the discrete rod configuration.</td></tr>
<tr><td colspan="2">Nodal discretization scheme without unit nodal director constraint</td><td>$\mathbb{R}^{6n_n}$</td><td>sparse, symmetric</td><td>$6(n_e+1)$</td></tr>
<tr><td colspan="5">Note that the nodal axial stress is not constrained to zero, however, nodal directors and director defined within elements live in different spaces: $\mathbb{R}^3$ and S^2, respectively.</td></tr>
<tr><td rowspan="3">Nodal discretization scheme with unit nodal director constraint</td><td>strong enforcement using Lagrange multiplier method</td><td>$(\mathbb{R}^3 \times S^2)^{n_n}$</td><td>sparse, non-symmetric</td><td>$7(n_e+1)$</td></tr>
<tr><td>strong enforcement with reduced equations using Lagrange multiplier method and nullspace method</td><td>$(\mathbb{R}^3 \times S^2)^{n_n}$</td><td>sparse, non-symmetric$^{(2)}$</td><td>$6(n_e+1)$</td></tr>
<tr><td>weak enforcement using penalty method</td><td>(strictly)$^{(3)}$ $\mathbb{R}^{6n_n}$</td><td>sparse, symmetric</td><td>$6(n_e+1)$</td></tr>
<tr><td colspan="5">Note that nodal axial stress is zero, however, the discrete director field lives in S^2 at any point of the discrete rod configuration.</td></tr>
<tr><td colspan="5">$^{(1)}$ When using the strong approach of outlier removal [4], global matrix multiplication is required. The multiplier is a constant matrix.
$^{(2)}$ Global matrix multiplication by the nullspace matrix is required. The multiplier is reassembled at each iteration and time step.
$^{(3)}$ With a penalty factor $\beta \to \infty$, the discrete solution space becomes $(\mathbb{R}^3 \times S^2)^{n_n}$.</td></tr>
</table>

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Poster presenter

T.-Hoa Nguyen
Postdoctoral Researcher
hoa.nguyen@uib.no
Web: thihoanguyen.com

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